Black-Hole versus Black-Body Thermodynamics

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According to the second law of thermodynamics, thermal interactions of material bodies lead to an increase in entropy. Black-hole thermodynamics has unknowingly repudiated this law of Nature.

The desideratum of black-hole thermodynamics is to relate the second law to the area theorem [1]. Just as the entropy will inevitably increase when material bodies at different temperatures are allowed to interact thermally, so too will the surface area of a black-hole when it undergoes any dynamical process [2]. The simplest relation, which does not contradict any conservation law, is a proportionality between the entropy $\tilde{S}$ and the surface area $\mathscr{A}$ [3],

$$\tilde{S} = \eta\, k\, \mathscr{A}/l^2, \tag{1}$$

where $l = \sqrt{\hbar G/c^3}$ is the Planck length. The constants k, $\hbar$, G and c are, respectively, Boltzmann's constant, Planck's constant divided by 2π, Newton's gravitational constant and the velocity of light in vacuum. The numerical factor, η, was subsequently fixed at $1/4$ by Hawking. Expression (1) is known as the Bekenstein-Hawking entropy, and it forms the basis for all subsequent developments in black-hole thermodynamics.

Planck placed a great deal of emphasis on the absolute nature of the entropy and the set of universal constants. The units of time, length, mass and temperature, which can be constructed from combinations of the fundamental set of constants, would truly be "independent of particular bodies or substances" and would "necessarily retain their significance for all times and for all cultures, including extraterrestrial and non human ones". Planck referred to them as "natural units" that would "retain their natural significance as long as the laws of gravitation and the propagation of light waves in vacuum, and the two laws of thermodynamics retain their validity" [4].

Division of the surface area by the square of the Planck length would, in addition to rendering $\tilde{S}/k$ a dimensionless quantity, allow the surface area to be measured in elementary units of l^2 in much the same way that the volume of phase space occupied by a system with f-degrees of freedom is measured in units of h^f. Since the system can never exist in a state of phase volume less than h^f and since the entropy is proportional to the logarithm of this ratio, it follows that the entropy continually shrinks to zero as the temperature is continually decreased. Nernst's theorem is thereby satisfied and the additive constant in the entropy expression has been fixed. This gives a meaning to the notion of an "absolute" value of the entropy rather than entropy differences which are physically measurable.

However, the division of the surface area by an elementary area cannot be vindicated by such an appeal to statistical mechanics since the entropy is not proportional to the logarithm of $\mathscr{A}/l^2$. Consequently, the division of $\mathscr{A}$ by l^2 will not merely fix the constant in the expression for the entropy but rather will fundamentally affect the value of the entropy as well as the entropy differences which are physically measurable. Moreover, the phase volume is proportional to the number of microstates, or "complexions" as Boltzmann referred to them, below a given value of the energy, its logarithm is a measure of the "thermodynamic probability" or likelihood of a macrostate corresponding to the number of complexions that are possible. In contrast, the surface area has a purely geometrical significance and, consequently, does not lend itself to the interpretation of an entropy as a measure of the number of complexions corresponding to a given macrostate; the larger the number, the more probable the state.

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In the particular case of a non-rotating and uncharged (Schwarzschild) black-hole, the entropy is

$$\tilde{S}(M) = \frac{4\pi k G}{c\hbar} M^2, \tag{2}$$

where M is the mass of the black-hole. The area theorem

$$(M_1 + M_2)^2 > M_1^2 + M_2^2 \tag{3}$$

states that when two black-holes collide, the area of the event horizon, proportional to the square of the irreducible mass M, must be greater than the sum of the areas of the individual event horizons prior to collison. Owing to inequality (3), for the entropy of the combined black-hole holds

$$\tilde{S}(M_1 + M_2) > \tilde{S}_1(M_1) + \tilde{S}_2(M_2), \tag{4}$$

which is referred to as the "superadditive" property of the entropy. This property supposedly replaces concavity when the latter no longer holds [5]. However, according to the second law, the entropy will increase only when the initial temperatures of the bodies are unequal. It is apparent that inequality (3), which forms the basis of the superadditivity of the entropy, (4), does not leave room for the eventuality that the entropy may be additive.

Since mass and energy are related by $E = M c^2$, one can *formally* define a temperature according to the prescription

$$\frac{\partial \tilde{S}}{\partial E} = \frac{8\pi k G}{c^5 \hbar} E = \frac{1}{T}. \tag{5}$$

Such a definition is, however, unacceptable thermodynamically, since the temperature is an *intensive* variable and not extensive like the energy or the mass. Proceeding formally, we may attempt to salvage the theory by enclosing the black-hole in a volume V greater than the Schwarzschild volume. Based solely on the extensivity of the entropy, an acceptable expression would be

$$S(E, V) = \frac{4\pi k G}{c^5 \hbar} \frac{E^2}{V}, \tag{6}$$

although it would raise havoc with inequality (3) that is used to determine the property of superadditivity (4). However, without a knowledge of the volume it would be impossible to do thermodynamics. Since its introduction in expression (6) in no way affects our conclusions, we will work with expression (6) rather than (2). The temperature is now defined as

$$\left(\frac{\partial S}{\partial E}\right)_V = \frac{8\pi k G}{c^5 \hbar} \frac{E}{V} = \frac{1}{T} \doteq k\beta. \tag{7}$$

Introducing it into the fundamental relation (6) results in the canonical form

$$S(E, V) = k[\beta(E) E - L(\beta(E), V)], \tag{8}$$

where

$$L(\beta, V) = \frac{c^5 \hbar}{16\pi G} V \beta^2 \tag{9}$$

is the Legendre transform of the entropy (6) with respect to the energy.

Since information is not lost in the Legendre transform (8) if one of the two functions is known to be either concave or convex, the other must be also [6]. From the fundamental relation (6) it is clear that the entropy is a *convex* function of the energy. In order to deduce the same from the canonical expression of the entropy, (8), we must know how β depends on the energy.

To this end, we introduce a generalized entropy

$$S[\beta, E, V] = k[\beta E - L(\beta, V)] \tag{10}$$

whose minimum value with respect to β will turn out to be the thermodynamic entropy (8). According to the implicit function theorem, we may solve

$$\frac{\partial S}{\partial \beta} = k\left[E - \frac{\partial L}{\partial \beta}\right] = 0, \tag{11}$$

since

$$\frac{\partial^2 S}{\partial \beta^2} \neq 0,$$

and obtain the curve $\beta(E)$ whose slope satisfies

$$\frac{\partial^2 S}{\partial E\, \partial \beta} + \frac{\partial^2 S}{\partial \beta^2} \frac{\partial \beta}{\partial E} = 0. \tag{12}$$

This curve is used to construct the function

$$S[\beta(E), E, V] = S(E, V). \tag{13}$$

And since $S(E, V)$ and $L(\beta, V)$ are Legendre transforms of one another, the curve $\beta(E)$ that results from (11) will necessarily be the same as the one that results from the second law (7).

When two black-holes collide – or at least are allowed to interact thermally – the energy of the combined black-hole is $E = E_1 + E_2$ and

$$L(\beta, V) = L_1(\beta, V_1) + L_2(\beta, V_2), \tag{14}$$

where $V = V_1 + V_2$. The introduction of the volume dependency in (6) was necessary for otherwise neither the entropy nor its Legendre transform would be extensive. Owing to the extensivity of energy and volume,

$$S[\beta, E, V] = S_1[\beta, E_1, V_1] + S_2[\beta, E_2, V_2]. \quad (15)$$

Observing that the functions $S_1[\beta, E_1, V_1]$ and $S_2[\beta, E_2, V_2]$ reach their *maxima* for $\beta = \beta_1(E_1)$ and $\beta = \beta_2(E_2)$, respectively, it follows that

$$S[\beta, E, V] \leqq S_1[\beta, E_1, V_1] + S_2[\beta, E_2, V_2] = S_1(E_1, V_1) + S_2(E_2, V_2). \quad (16)$$

Inequality (16) asserts that the entropy of the combined system, obtained when two black-holes are allow to interact thermally, is never greater than the sum of the entropies of the individual black-holes. The entropy has been defined as

$$S(E, V) = \max_{\beta}[\beta E - L(\beta, V)] = \beta(E) E - L(\beta(E), V), \quad (17)$$

which is a *convex* function of E. For a fixed value of E, $S[\beta, E, V]$ is a *concave* function of β. This definition is much more suitable for the Kullback-Leibler information statistic [7] rather than the entropy.

Due to the convexity property of the entropy, it violates the thermodynamic stability criterion that the heat capacity be positive. If such a system did exist, as black-hole thermodynamicists claim, it could be used to construct a perpetual motion machine of the second type since the extraction of heat would lead to an increase in the temperature of such a body. In Eddington's words, "under classical law [stars] seemed to be heading towards an intolerable situation – the star could not stop losing heat, but it would have insufficient energy to be able to cool down" [8]. Yet, he didnot seem to connect this with his earlier statement that "if your theory is found to be against the second law of thermodynamics I can give you no hope; there is nothing for it but to collapse in deepest humiliation" [9].

The reason for the superadditivity property (4) is that when we have a λ-fold increase in the mass, with $\lambda > 1$, $(\lambda M)^2 > \lambda M^2$. Consequently, the entropy of the final state would always be greater than λ times the entropy of the initial state. Regretfully, the entropy of such a system has lost the property of extensivity. The entropy must always be extensive if the variables upon which it depends in the fundamental relation are extensive. An alteration of the extensive property of the entropy would also require a similar alteration of the first law! The irreversible tendency of the entropy to increase is a result of the interaction of bodies that are brought into thermal contact in which their temperatures are allowed to adjust themselves thereby leading to a new, homogeneous state of thermal equilibrium.

The entropy, being a monotonic function of the energy, must always increase or always decrease [10]. It suffices to take a known physical system and show that the entropy increases. Black-holes will then be out-lawed as being non-physical or there is a basic flaw in the identification of the entropy with the area of a black-hole, (1).

The fundamental relation for black-body radiation is

$$S(E, V) = \tfrac{4}{3} b^{1/4} E^{3/4} V^{1/4}, \quad (18)$$

in the entropy representation, where V is the volume of the cavity and $c\,b/4$ is the Stefan-Boltzmann constant. The entropy is not a function of the number of photons since they are not conserved and hence there is no closure condition [11]. Nonetheless, (18) is not the true fundamental relation since it does not correspond to an error law for the energy. The true fundamental relation was found by Planck in his study of black-body radiation, and it consists of the expression for the entropy as a logarithmic function of the number of photons in a given frequency interval of the electromagnetic field. This fundamental relation of the entropy leads to a law of error, the negative binomial distribution, for which the average number of particles is the most probable value [12].

If the energy and volume were to undergo a λ-fold increase, the entropy,

$$\lambda S(E, V) = S(\lambda E, \lambda V),$$

would increase λ times *for any* λ. The entropy of black-body radiation is, therefore, extensive; no increase in energy or volume can destroy its extensivity since both these quantities are, themselves, extensive.

The temperature is defined as

$$\left(\frac{\partial S}{\partial E}\right)_V = \left(\frac{b\,V}{E}\right)^{1/4} \doteq k\beta. \quad (19)$$

Proceeding as before, (19) is introduced into the fundamental relation (18) to obtain the canonical form of the entropy (8), where the Legendre dual function is

$$L(\beta, V) = -\frac{b\,V}{3\,k^4 \beta^3}. \quad (20)$$

The generalized entropy, (10), is a saddle function which is strictly *convex* in β and weakly *concave* in E [13]. Owing to the strict convexity of the generalized entropy with respect to β, we can solve (11) for the curve $\beta(E)$ whose corresponding path on the saddle function has the height (13). By the chain rule,

$$\frac{\partial^2}{\partial E^2} S(E, V) = -\left(\frac{\partial^2}{\partial E\,\partial\beta} S[\beta, E, V]\right)^2 \cdot\left(\frac{\partial^2}{\partial\beta^2} S[\beta, E, V]\right)^{-1} < 0, \quad (21)$$

it follows that the thermodynamic entropy, (13), is a strictly concave function of the energy.

Owing to the extensivity of energy and volume, the generalized entropy is also extensive. When two black-bodies are brought into thermal contact, having the same value of β, the entropy will be given by (15). However, since $S_1[\beta, E_1, V_1]$ and $S_2[\beta, E_2, V_2]$ reach a minimum at $\beta = \beta_1(E_1)$ and $\beta = \beta_2(E_2)$, respectively, being solutions to the stationarity condition (11), it follows that

$$S[\beta, E, V] \geqq S(\beta(E_1), E_1, V_1) + S_2(\beta(E_2), E_2, V_2) = S_1(E_1, V_1) + S_2(E_2, V_2). \quad (22)$$

In words, (22) states that the entropy of a composite system, formed by the thermal interaction of two previously isolated systems, can never be inferior to the sum of the entropies of the two subsystems. It is inequality (22) and *not* (16) that contains the essence of the second law.

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